Problem 3.35

Consider the wave function

$$\Psi(x,0) = \begin{cases} \frac{1}{\sqrt{2n\lambda}} e^{i2\pi x/\lambda}, & -n\lambda < x < n\lambda, \\ 0, & \text{otherwise,} \end{cases}$$

where n is some positive integer. This function is purely sinusoidal (with wavelength λ) on the interval $-n\lambda < x < n\lambda$, but it still carries a *range* of momenta, because the oscillations do not continue out to infinity. Find the momentum space wave function $\Phi(p,0)$. Sketch the graphs of $|\Psi(x,0)|^2$ and $|\Phi(p,0)|^2$, and determine their widths, w_x and w_p (the distance between zeros on either side of the main peak). Note what happens to each width as $n \to \infty$. Using w_x and w_p as estimates of Δx and Δp, check that the uncertainty principle is satisfied. *Warning:* If you try calculating σ_p, you're in for a rude surprise. Can you diagnose the problem?

Solution

Calculate $|\Psi(x,0)|^2$, the probability distribution for the particle's position at $t = 0$.

$$|\Psi(x,0)|^2 = \Psi^*(x,0)\Psi(x,0) = \begin{cases} \left(\frac{1}{\sqrt{2n\lambda}} e^{-i2\pi x/\lambda}\right)\left(\frac{1}{\sqrt{2n\lambda}} e^{i2\pi x/\lambda}\right) & \text{if } -n\lambda < x < n\lambda \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{|\Psi(x,0)|^2 = \begin{cases} \frac{1}{2n\lambda} & \text{if } -n\lambda < x < n\lambda \\ 0 & \text{otherwise} \end{cases}}$$

Check that it's normalized.

$$\int_{-\infty}^{\infty} |\Psi(x,0)|^2 \, dx = \int_{-n\lambda}^{n\lambda} \frac{1}{2n\lambda} \, dx = \frac{1}{2n\lambda} \int_{-n\lambda}^{n\lambda} dx = \frac{1}{2n\lambda}(2n\lambda) = 1$$

Now obtain $\Phi(p,0)$, the momentum-space wave function at $t = 0$, by taking the Fourier transform of $\Psi(x,0)$.

$$\begin{aligned} \Phi(p,0) &= \mathscr{F}\{\Psi(x,0)\} \\ &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \Psi(x,0) \, dx \\ &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-n\lambda}^{n\lambda} e^{-ipx/\hbar} \frac{1}{\sqrt{2n\lambda}} e^{i2\pi x/\lambda} \, dx \\ &= \frac{1}{2\sqrt{n\pi\hbar\lambda}} \int_{-n\lambda}^{n\lambda} \exp\left[i\left(\frac{2\pi}{\lambda} - \frac{p}{\hbar}\right)x\right] dx \\ &= \frac{1}{2\sqrt{n\pi\hbar\lambda}} \int_{-n\lambda}^{n\lambda} \exp\left[i\left(\frac{2\pi\hbar - p\lambda}{\hbar\lambda}\right)x\right] dx \end{aligned}$$

Note that h can't be substituted for $p\lambda$ (Equation 1.39 on page 19) because, as Mr. Griffiths said, this sinusoidal wave in $\Psi(x,0)$ does not extend indefinitely.

Evaluate the integral and simplify the result.

$$\begin{aligned}
\Phi(p,0) &= \frac{1}{2\sqrt{n\pi\hbar\lambda}} \cdot \frac{\hbar\lambda}{i(2\pi\hbar - p\lambda)} \exp\left[i\left(\frac{2\pi\hbar - p\lambda}{\hbar\lambda}\right)x\right]\Bigg|_{-n\lambda}^{n\lambda} \\
&= \frac{1}{2i}\sqrt{\frac{\hbar\lambda}{n\pi}} \cdot \frac{1}{2\pi\hbar - p\lambda}\left\{\exp\left[i\left(\frac{2\pi\hbar - p\lambda}{\hbar\lambda}\right)n\lambda\right] - \exp\left[-i\left(\frac{2\pi\hbar - p\lambda}{\hbar\lambda}\right)n\lambda\right]\right\} \\
&= \frac{1}{2i}\sqrt{\frac{\hbar\lambda}{n\pi}} \cdot \frac{1}{2\pi\hbar - p\lambda}\left(e^{2i\pi n}e^{-ip\lambda n/\hbar} - e^{-2i\pi n}e^{ip\lambda n/\hbar}\right) \\
&= \frac{1}{2i}\sqrt{\frac{\hbar\lambda}{n\pi}} \cdot \frac{1}{p\lambda - 2\pi\hbar}\left(e^{ip\lambda n/\hbar} - e^{-ip\lambda n/\hbar}\right) \\
&= \frac{1}{2i}\sqrt{\frac{\hbar\lambda}{n\pi}} \cdot \frac{1}{p\lambda - 2\pi\hbar}\left(2i\sin\frac{p\lambda n}{\hbar}\right) \\
&= \sqrt{\frac{\hbar\lambda}{n\pi}} \cdot \frac{\sin\frac{p\lambda n}{\hbar}}{p\lambda - 2\pi\hbar}
\end{aligned}$$

Now determine $|\Phi(p,0)|^2$, the probability distribution for the particle's momentum at $t = 0$.

$$\boxed{|\Phi(p,0)|^2 = \Phi^*(p,0)\Phi(p,0) = \frac{\hbar\lambda}{n\pi} \cdot \frac{\sin^2\frac{p\lambda n}{\hbar}}{(p\lambda - 2\pi\hbar)^2} = \frac{\lambda}{n\pi\hbar} \cdot \frac{\sin^2\frac{p\lambda n}{\hbar}}{\left(\frac{p\lambda}{\hbar} - 2\pi\right)^2}}$$

Check that it's normalized.

$$\int_{-\infty}^{\infty} |\Phi(p,0)|^2\,dp = \frac{\hbar\lambda}{n\pi}\int_{-\infty}^{\infty} \frac{\sin^2\frac{p\lambda n}{\hbar}}{(p\lambda - 2\pi\hbar)^2}\,dp$$

Make the following substitution.

$$p\lambda = 2\pi\hbar u \qquad \rightarrow \qquad \frac{p\lambda n}{\hbar} = 2\pi n u$$

$$dp = \frac{2\pi\hbar}{\lambda}\,du$$

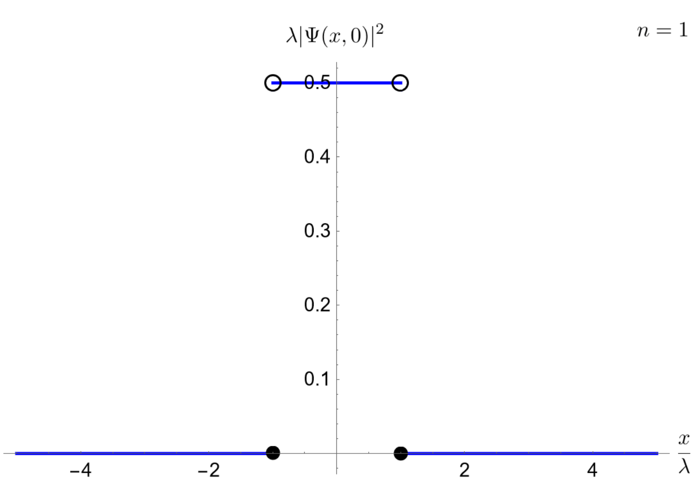
Consequently,

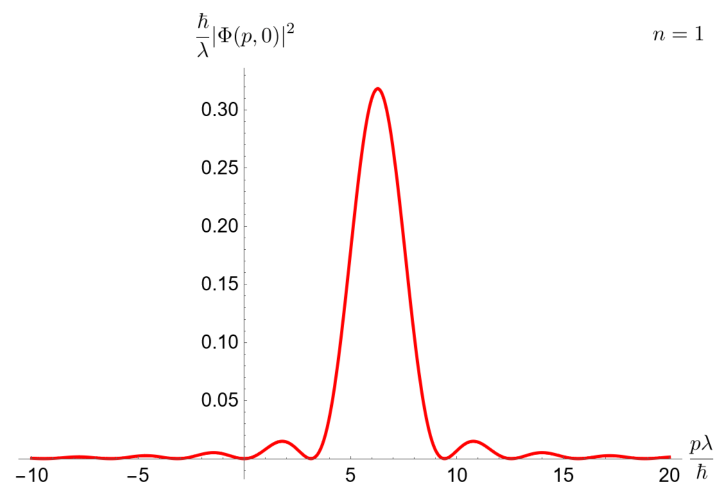
$$\begin{aligned}
\int_{-\infty}^{\infty} |\Phi(p,0)|^2\,dp &= \frac{\hbar\lambda}{n\pi}\int_{-\infty}^{\infty} \frac{\sin^2 2\pi n u}{(2\pi\hbar u - 2\pi\hbar)^2}\left(\frac{2\pi\hbar}{\lambda}\,du\right) \\
&= \frac{\hbar\lambda}{n\pi} \cdot \frac{1}{4\pi^2\hbar^2} \cdot \frac{2\pi\hbar}{\lambda}\int_{-\infty}^{\infty} \frac{\sin^2 2\pi n u}{(u-1)^2}\,du \\
&= \frac{1}{2n\pi^2}\int_{-\infty}^{\infty} \frac{\sin^2 2\pi n u}{(u-1)^2}\,du.
\end{aligned}$$

Substitute $v = u - 1$ $(du = dv)$ and then use the fact that sine is 2π-periodic.

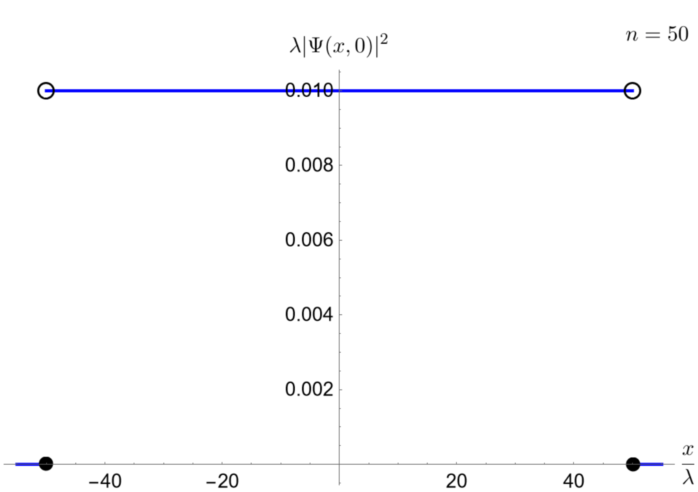
$$\begin{aligned}
\int_{-\infty}^{\infty} |\Phi(p,0)|^2 \, dp &= \frac{1}{2n\pi^2} \int_{-\infty}^{\infty} \frac{\sin^2 2\pi n(v+1)}{v^2} \, dv \\
&= \frac{1}{2n\pi^2} \int_{-\infty}^{\infty} \frac{\sin^2(2\pi n v + 2\pi n)}{v^2} \, dv \\
&= \frac{1}{2n\pi^2} \int_{-\infty}^{\infty} \frac{\sin^2 2\pi n v}{v^2} \, dv \\
&= \frac{1}{2n\pi^2}(2n\pi^2) \\
&= 1
\end{aligned}$$

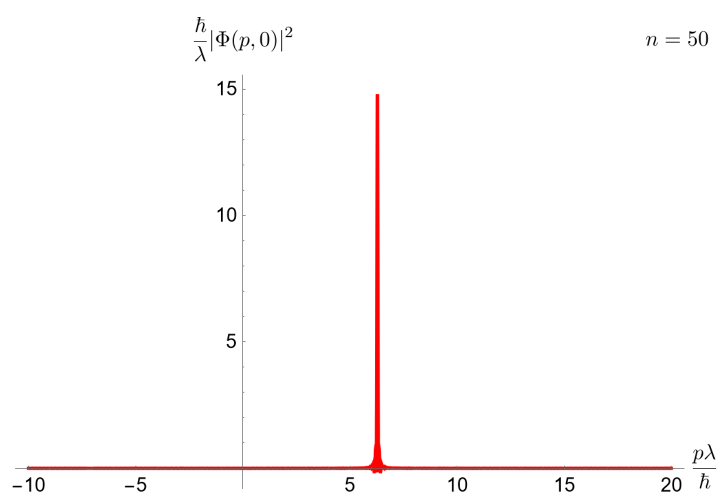
Below is a side-by-side comparison of the probability distributions at $t = 0$ for $n = 1$.





Below is a side-by-side comparison of the probability distributions at $t = 0$ for $n = 50$.





Notice that increasing n increases the uncertainty in the particle's position and decreases the uncertainty in the particle's momentum.

Here we use the width (from zero to zero) of the highest peak in the probability distribution to quantify the uncertainty. For the position, the highest peak occurs for

$$-n \leq \frac{x}{\lambda} \leq n$$

$$-n\lambda \leq x \leq n\lambda.$$

As a result, the width for position is

$$\boxed{w_x = n\lambda - (-n\lambda) = 2n\lambda.}$$

For the momentum, observe that the absolute maximum occurs when $p\lambda/\hbar = 2\pi$. The numerator here is $\sin^2 2\pi n$. The closest zeros on either side of this absolute maximum occur at the next and previous multiples of π. The highest peak, then, occurs for

$$(2n-1)\pi \leq \frac{np\lambda}{\hbar} \leq (2n+1)\pi$$

$$(2n-1)\pi\hbar \leq np\lambda \leq (2n+1)\pi\hbar$$

$$\frac{(2n-1)\pi\hbar}{n\lambda} \leq p \leq \frac{(2n+1)\pi\hbar}{n\lambda}.$$

Consequently, the width for momentum is

$$\boxed{w_p = \frac{(2n+1)\pi\hbar}{n\lambda} - \frac{(2n-1)\pi\hbar}{n\lambda} = \frac{2\pi\hbar}{n\lambda}.}$$

As expected,

$$\lim_{n\to\infty} w_x = \infty \quad \text{and} \quad \lim_{n\to\infty} w_p = 0.$$

Now take the product of w_x and w_p and check that Heisenberg's uncertainty principle is satisfied at $t = 0$.

$$w_x w_p = (2n\lambda)\left(\frac{2\pi\hbar}{n\lambda}\right) = 4\pi\hbar \geq \frac{\hbar}{2}.$$

Calculating σ_p as usual,

$$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2},$$

isn't useful here because $\langle p^2\rangle$ is infinite regardless of which wave function is used to calculate it. Using the momentum-space wave function $\Phi(p, 0)$ leads to a divergent integral.

$$\langle p^2\rangle = \langle \Phi \,|\, \hat{p}^2 \,|\, \Phi\rangle = \int_{-\infty}^{\infty} \Phi^*(p,0)p^2\Phi(p,0)\,dp = \int_{-\infty}^{\infty} p^2|\Phi(p,0)|^2\,dp = \frac{\hbar\lambda}{n\pi}\int_{-\infty}^{\infty} \frac{p^2 \sin^2 \frac{p\lambda n}{\hbar}}{(p\lambda - 2\pi\hbar)^2}\,dp = \infty$$

Using the position-space wave function $\Psi(x, 0)$ instead isn't any better because it's more complicated than it seems, and it has to be differentiated twice.

$$\Psi(x,0) = \frac{1}{\sqrt{2n\lambda}} e^{i2\pi x/\lambda}\bigg[\theta(x+n\lambda) - \theta(x-n\lambda)\bigg], \quad -\infty < x < \infty$$

The problem comes from the fact that there are Heaviside functions, which in turn are due to the discontinuities at $x = \pm n\lambda$.